Conformal invariants of hypersurfaces

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Part I

Motivation

Riemannian hypersurfaces

A Riemannian oriented manifold (M, g), $\dim M \geq 3$, $\mathrm{T}M =: \mathcal{E}^a$

A oriented embedded hypersurface $\Sigma \subset M$, $\mathrm{T}\Sigma =: \overline{\mathcal{E}}^a$

The pullback bundle $\mathrm{T}M|_\Sigma =: \underline{\mathcal{E}}^a \equiv \overline{\mathcal{E}}^a \oplus \mathcal{N}^a$

The pullback connection $\underline{\nabla}_a : \underline{\mathcal{E}}^b \to \overline{\mathcal{E}}_a \otimes \underline{\mathcal{E}}^b$

Metric $g \equiv g|_\Sigma$ and connection $\underline{\nabla}$ in $\underline{\mathcal{E}}^a$ are termed *ambient*

The induced metric $\bar{g}$ and the L.-C. connection $\overline{\nabla} := \nabla^{\bar{g}}$

Metric $\bar{g}$ and connection $\overline{\nabla}$ in $\overline{\mathcal{E}}^a$ are termed *intrinsic*

The unit normal $\mathrm{N}^a \in \underline{\mathcal{E}}^a$ (agrees with the orientations)

The tangential projection $\Pi_a{}^b = \delta_a{}^b - \mathrm{N}_a \mathrm{N}^b$

The shape tensor $\mathrm{L}_a{}^b = \underline{\nabla}_a \mathrm{N}^b$; the mean curvature $\mathrm{H} = \frac{1}{\dim \Sigma} \mathrm{L}_a{}^a$

Conformal hypersurfaces

Assumptions as before, but now (M, c), a conformal manifold

Conformal structure $c = [g]$, $\widehat{g} \sim g \Leftrightarrow \widehat{g} = \Omega^2 g$, $\Omega > 0$

The induced conformal structure $\bar{c} = [\bar{g}]$ on Σ

Conformal densities $\mathcal{E}[w]$ agree: $\overline{\mathcal{E}}[w] = \mathcal{E}[w]|_\Sigma$

Conformal metric $\mathbf{g}_{ab} \in \mathcal{E}^+_{(ab)}[2]$ on M, equiv. to c

The unit normal is viewed as $\mathbf{N}^a \in \underline{\mathcal{E}}^a[-1]$ for $\mathbf{g}_{ab}\mathbf{N}^a\mathbf{N}^b = 1$

The shape tensor as a tensor-density $\mathbf{L}_{ab} \in \overline{\mathcal{E}}_{(ab)}[1]$

The mean curvature as a density $\mathbf{H} \in \overline{\mathcal{E}}[-1]$

$\widehat{\mathbf{L}_{ab}} = \mathbf{L}_{ab} + \Upsilon^c \mathbf{N}_c \bar{\mathbf{g}}_{ab}$, where $\Upsilon_a = \nabla_a \log \Omega$; $\widehat{\mathbf{H}} = \mathbf{H} + \Upsilon^c \mathbf{N}_c$

The *umbilicity* tensor $\mathring{\mathbf{L}}_{ab} = \mathbf{L}_{ab} - \mathbf{H}\mathbf{g}_{ab}$ is invariant $\widehat{\mathring{\mathbf{L}}_{ab}} = \mathring{\mathbf{L}}_{ab}$

The Willmore invariant

Let $\dim M = 3$, so $\dim \Sigma = 2$, and $\mathrm{d}\Sigma$ has weight 2.

The Willmore functional

$$\int_\Sigma |\mathring{\mathbf{L}}|^2 \mathrm{d}\Sigma$$

is conformally invariant.

The Euler–Lagrange equation

$$\underbrace{\overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + \mathrm{Ric}^{ab} \mathring{\mathbf{L}}_{ab} + \mathbf{H} |\mathring{\mathbf{L}}|^2} = 0$$

This is an example of what we want to mean by a conformal invariant of hypersurfaces.

More examples

Let $\dim \Sigma = 4$.
The intrinsic Weyl tensor $\overline{\mathrm{W}}$, the ambient Weyl tensor W.
Here $\mathrm{R}_{ab}{}^{cd} = \mathrm{W}_{ab}{}^{cd} + 4\delta_{[a}{}^{[c}\mathsf{P}_{b]}{}^{d]}$ for $\mathrm{R}_{ab}{}^{c}{}_{d}V^{d} = 2\nabla_{[a}\nabla_{b]}V^{c}$

The E.–L. equation for $\int_\Sigma |\overline{\mathrm{W}}|^2 \mathrm{d}\Sigma$ is (where $\overline{\mathrm{B}}_{ab}$ is the intrinsic Bach tensor)

$$\overline{\mathrm{B}}^{ab}\mathring{\mathbf{L}}_{ab} = 0$$

The E.–L. equation for $\int_\Sigma |\mathrm{W}|^2 \mathrm{d}\Sigma$

$$\left(\mathbf{N}^{e}\nabla_{e}\mathrm{W}^{abcd} + 2\mathbf{H}\mathrm{W}^{abcd}\right)\mathrm{W}_{abcd} = 0$$

There are other examples; for instance, when $\dim \Sigma > 3$,

$$\Pi_{a}{}^{a'}\Pi_{b}{}^{b'}\mathsf{P}_{a'b'} - \overline{\mathsf{P}}_{ab} + \mathbf{H}\mathring{\mathbf{L}}_{ab} + \frac{\mathbf{H}^2}{2}\bar{\mathbf{g}}_{ab}$$

Part II

Metric invariants

Metric invariants

Definition

A **scalar local metric invariant** is a function $P(g)\colon M \to \mathbb{R}$ such that

1. in any choice (U, x^i) of local coordinates $P(g)$ is represented by a universal polynomial in variables $\partial_{k^1} \ldots \partial_{k^r} g_{ij}$, for $r \geq 0$ and $(\det g)^{-1}$
2. for any local diffeomorphism $\varphi\colon M \to M$: $P(\varphi^* g) = \varphi^* P(g)$

Example

Scalar curvature $\mathrm{Scal} = g^{jl}\left(\Gamma^m_{il}\Gamma^i_{jm} - \Gamma^m_{jl}\Gamma^i_{im} + \partial_j \Gamma^i_{il} - \partial_i \Gamma^i_{jl}\right)$

Tensor-valued metric invariants are defined similarly if we allow values of $P(g)$ in tensor bundles, e.g. Riemannian curvature R , Ricci tensor Ric etc

The Levi-Civita connection ∇ is an example of a metric invariant differential operator

Metric Weyl invariants

Definition

A **metric Weyl invariant** is a linear combination of (complete or partial) contractions (contr) of one of the forms

$$\text{contr}\left(g \otimes \cdots \otimes g \otimes \mathrm{R}^{(s_1)} \otimes \cdots \otimes \mathrm{R}^{(s_r)}\right) \qquad \text{"even"}$$
$$\text{contr}\left(\epsilon \otimes g \otimes \cdots \otimes g \otimes \mathrm{R}^{(s_1)} \otimes \cdots \otimes \mathrm{R}^{(s_r)}\right) \qquad \text{"odd"}$$

where

$$\mathrm{R}^{(s)}_{abcdp^1\ldots p^s} := \nabla_{p^1} \ldots \nabla_{p^s} \mathrm{R}_{abcd}$$

for $s \geq 0$, g is the Riemannian metric, ϵ is the Riemannian volume form.

The complete contractions yield scalar invariants, the partial provide tensor-valued.

Similarly one can define metric Weyl invariant differential operators, e.g.

$\Delta f = \nabla^a \nabla_a f$, the Laplacian

Classical Invariant Theory

Theorem

All local metric invariants are obtained as metric Weyl invariants.

Same is true for metric invariant differential operators.

Hypersurface metric invariants

Let Σ is an oriented hypersurface in an oriented Riemannian manifold (M, g), and s is an **oriented** defining function of Σ.

Definition (Gover-Waldron)

A function $P(s, g)$ on M with values (in a tensor bundle on M) given in any local coordinate system U, x^i by universal polynomial expressions in $(\det g)^{-1}$, $\partial_{p^1} \dots \partial_{p^k} g_{ij}$ for $k \geq 0$, $|\mathrm{d}s|^{-1}$, and $\partial_{q^1} \dots \partial_{q^l} s$ for $l \geq 0$, is called a hypersurface metric **pre-invariant** if for any local diffeomorphism $\varphi \colon M \to M$

$$P(\varphi^* s, \varphi^* g) = \varphi^* P(s, g)$$

A hypersurface metric pre-invariant $P(s, g)$ is called a **hypersurface metric invariant** if its *restriction* onto Σ is independent of the choice of compatibly oriented defining function:

$$P(s', g) = P(s, g) \text{ along } \Sigma$$

Part III

Conformal Invariant Theory

Conformal Invariants

Some of the metric invariants transform under conformal rescalings in a particularly simple way.

Definition

A **local conformal invariant with conformal weight** w is a local metric invariant $P(g)$ with values in a weighted (tensored with $\mathcal{E}[w]$) tensor bundle such that $\widehat{P} = P$, i.e.

$$P(\Omega^2 g) = P(g)$$

Examples

The Weyl tensor $\widehat{\mathrm{W}_{ab}{}^{c}{}_{d}} = \mathrm{W}_{ab}{}^{c}{}_{d}$ is a tensor-valued conformal invariant of zero weight

The square of the length of the Weyl tensor $\widehat{|\mathrm{W}|^2} = |\mathrm{W}|^2 \in \mathcal{E}[-4]$

The standard conformal tractor bundle

A vector bundle $\mathcal{T}^A$ which in any choice g of a metric from the conformal class c is represented as

$$\mathcal{T}^A \stackrel{g}{=} \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$$

$[V^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}$ is replaced by $[V^A]_{\widehat{g}} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_b \mu^b - \frac{\Upsilon_b \Upsilon^b}{2}\sigma \end{pmatrix}$ for $\widehat{g} = \Omega^2 g$

The tractor metric $h_{AB} U^A V^B$ defined by

$$U_A V^A \stackrel{g}{=} \begin{pmatrix} \rho' & \mu'_a & \sigma' \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} = \rho'\sigma + \mu'_a \mu^a + \sigma'\rho$$

A compatible ($\nabla_a h_{BC} = 0$) conformally invariant connection:

$$\nabla_b V^A \stackrel{g}{=} \begin{pmatrix} \nabla_b \sigma - \delta_b{}^a \mu_a \\ \nabla_b \mu^a + \delta_b{}^a \rho + \mathrm{P}_b{}^a \sigma \\ \nabla_b \rho - \mathrm{P}_b{}^a \mu_a \end{pmatrix}$$

Tractor Projectors

Section $X^A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is invariant, whereas the sections $Y^A \stackrel{g}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Z^A{}_b \stackrel{g}{=} \begin{pmatrix} 0 \\ \delta^a{}_b \\ 1 \end{pmatrix}$ do rescale: $\begin{cases} \widehat{Y^A} = Y^A - Z^A{}_b \Upsilon^b - X^A \frac{\Upsilon_b \Upsilon^b}{2} \\ \widehat{Z^A{}_b} = Z^A{}_b + X^A \Upsilon_b \\ \widehat{X^A} = X^A \end{cases}$

Any tractor $V^A \in \mathcal{T}^A$ can be represented in a scale $g \in c$ using the **standard tractor projectors** $Y^A \in \mathcal{T}^A[-1]$, $Z^A{}_b \in \mathcal{T}^A \otimes \mathcal{E}_a[1]$ and $X^A \in \mathcal{T}^A[1]$ as

$$\boxed{V^A \stackrel{g}{=} Y^A \sigma + Z^A{}_b \mu^b + X^A \rho}$$

Significantly simplify calculations: the tractor connection is given by

$$\begin{cases} \nabla_b Y_A \stackrel{g}{=} Z_A{}^a \mathrm{P}_{ab} \\ \nabla_b Z_A{}^a \stackrel{g}{=} -Y_A \mathbf{g}_{ba} - X_A \mathrm{P}_{ba} \\ \nabla_b X_A = Z_A{}^a \mathbf{g}_{ba} \stackrel{g}{=} \end{cases}$$

The tractor metric: $h_{AB} \stackrel{g}{=} Y_A X_B + \mathbf{g}_{ab} Z_A{}^b Z_A{}^b + X_A Y_B$

Tractor operators

The tractor pre-D operator

The operator $\boxed{\mathfrak{D}_A f \overset{g}{=} Y_A w f + Z_A{}^a \nabla_a f}$ acting on $\mathcal{E}[w]$ rescales:

$$\widehat{\mathfrak{D}_A f} = \mathfrak{D}_A f + X_A \left(\Upsilon \cdot \nabla f + \frac{|\Upsilon|^2}{2} w\, f \right) \qquad (*)$$

The tractor double-D operator

This immediately get conformally invariant $\boxed{\mathbb{D}_{AP} f \overset{g}{=} 2X_{[P}\mathfrak{D}_{A]} f}$

The Thomas-D operator

Comparing $(*)$ with $\widehat{\Box f} = \Box f + (m + 2w - 2) \left(\Upsilon \cdot \nabla f + \frac{|\Upsilon|^2}{2} w\, f \right)$ for $\Box f = \Delta f + J w f$ we recover conformally invariant operator:

$$\boxed{\mathrm{D}_A f \overset{g}{=} (m + 2w - 2)\mathfrak{D}_A f - X_A \Box f}$$

$(m = \dim M)$

Weyl tractor

The curvature of the tractor connection

$$\Omega_{ab}{}^{C}{}_{D}V^{D} = 2\nabla_{[a}\nabla_{b]}V^{C}$$

Explicitly, using the Cotton tensor $\mathrm{Y}_{abd} := 2\,\nabla_{[a}\mathrm{P}_{b]d}$,

$$\Omega_{abCD} \overset{g}{=} Z_{C}{}^{c}Z_{D}{}^{d}\mathrm{W}_{abcd} - 2X_{[C}Z_{D]}{}^{d}\mathrm{Y}_{abd}$$

Define **the Weyl tractor** as

$$\boxed{\mathrm{W}_{ABCD} := \frac{3}{m-2}\mathrm{D}^{A'}X_{[A'}\Omega_{AB]CD}}$$

where

$$\Omega_{ABCD} \overset{g}{=} Z_{A}{}^{a}Z_{B}{}^{b}\Omega_{abCD}$$

Explicitly, using the Bach tensor $\mathrm{B}_{bd} := \nabla^{a}\mathrm{Y}_{abd} + \mathrm{P}^{ac}\mathrm{W}_{abcd}$, we have

$$\mathrm{W}_{ABCD} \overset{g}{=} (m-4)\Big(Z_{A}{}^{a}Z_{B}{}^{b}Z_{C}{}^{c}Z_{D}{}^{d}\mathrm{W}_{abcd} - 2Z_{A}{}^{a}Z_{B}{}^{b}X_{[C}Z_{D]}{}^{d}Y_{abd} \\ - 2X_{[A}Z_{B]}{}^{b}Z_{C}{}^{c}Z_{D}{}^{d}\mathrm{Y}_{cdb}\Big) + 4X_{[A}Z_{B]}{}^{b}X_{[C}Z_{D]}{}^{d}\mathrm{B}_{bd}$$

Conformal Weyl invariants

Using D_A and W_{ABCD} we can imitate Weyl invariants for the conformal structure

Definition (A.R.Gover)

A **conformal Weyl invariant** is a linear combination of (partial or complete) tractor contractions (contr) of one of the forms

$$\begin{aligned} &\text{contr}\left(h \otimes \cdots \otimes h \otimes \mathrm{W}^{(s_1)} \otimes \cdots \otimes \mathrm{W}^{(s_r)}\right) && \text{"even"} \\ &\text{contr}\left(\eta \otimes h \otimes \cdots \otimes h \otimes \mathrm{W}^{(s_1)} \otimes \cdots \otimes \mathrm{W}^{(s_r)}\right) && \text{"odd"} \end{aligned}$$

where

$$\mathrm{W}^{(s)}_{ABCDP^1\ldots P^s} := \mathrm{D}_{P^1} \ldots \mathrm{D}_{P^s} \mathrm{W}_{ABCD}$$

for $s \geq 0$, h is the tractor metric, η is the volume tractor form.

Conformal Quasi-Weyl Invariants

Construction:

Step 1 (for $n > 3$)

a) Form $\mathbb{D}_{PP'} \dots \mathbb{D}_{QQ'} \mathrm{C}_{A'ABCDD'}$, b) contract some tractor indices and c) take its symmetric trace-free part

Here $\mathrm{C}_{A'ABCDD'} := 9 X_{[A'} Z_A{}^a Z_{B]}{}^b Z_{[C}{}^c Z_D{}^d X_{D']} \mathrm{W}_{abcd}$

There is a modification of this step for $n = 3$

Step 2

Manipulate to $X_{(A} \dots X_E J_{F \dots)\circ}$

Step 3

Using the tractor-D operator "eliminate" some indices, e.g. $\mathrm{D}^F J_{F \dots}$

Definition (A.R.Gover)

Linear combinations of so obtained tractors (or densities) are called *conformal quasi-Weyl invariants*

Part IV

Constructions of hypersurface conformal invariants

Hypersurface Conformal Invariants

Definition

A **hypersurface conformal invariant** is a hypersurface metric invariant $P(s, g)$, which under the rescaling $\widehat{g} = \Omega^2 g$ of the metric in the ambient conformal class satisfies

$$P(s, \Omega^2 g) = P(s, g)$$

along the hypersurface.

Examples

The unit conormal $\mathbf{N}_a$, the ambient conformal metric $\mathbf{g}_{ab}$, the intrinsic conformal metric $\bar{\mathbf{g}}_{ab}$, the umbilicity tensor $\mathring{\mathbf{L}}_{ab}$, its square $|\mathring{\mathbf{L}}|^2$, the ambient Weyl tensor W_{abcd} and its square $|\mathrm{W}|^2$, the intrinsic Weyl tensor $\overline{\mathrm{W}}_{abcd}$ and its square $|\overline{\mathrm{W}}|^2$.

Hypersurface Tractors

The ambient tractor bundle $\underline{\mathcal{T}}^A := \mathcal{T}^A|_\Sigma$ has the metric h_{AB} and the pullback connection $\underline{\nabla}^{\mathcal{T}}_a$ called the **ambient tractor connection**.

The **normal tractor** is $\boxed{\mathrm{N}^A \stackrel{g}{=} Z^A{}_a \mathbf{N}^a - X^A \mathbf{H}} \in \underline{\mathcal{T}}^A$

It is invariant $\widehat{\mathrm{N}^A} = \mathrm{N}^A$ and has unit length $\mathrm{N}_A \mathrm{N}^A = 1$

The orthogonal decomposition $\underline{\mathcal{T}}^A = \overline{\mathcal{T}}^A \oplus \mathcal{N}^A$

The intrinsic tractor bundle $\overline{\mathcal{T}}^A$ is generated by the intrinsic tractor projectors

$$\bar{Y}^A := Y^A + Z^A{}_a \mathbf{N}^a \mathbf{H} - \frac{\mathbf{H}^2}{2} X^A$$
$$\bar{Z}^A{}_a := Z^A{}_b \Pi^b{}_a$$
$$\bar{X}^A := X^A$$

Hypersurface Tractor Operators

Let $\dim \Sigma = n = m - 1 \geq 2$

Intrinsic pre-D operator $\overline{\mathfrak{D}}_A f := \bar{Y}_A w f + \bar{Z}_A{}^a \overline{\nabla}_a f$
Intrinsic double-D operator $\overline{\mathbb{D}}_{AP} := 2\bar{X}_{[P}\overline{\mathfrak{D}}_{A]} f$
Intrinsic Thomas-D operator $\overline{\mathrm{D}}_{AP} := (n + 2w - 2)\overline{\mathfrak{D}}_A f - \bar{X}_A \overline{\Box} f$

These operators are defined on $\overline{\mathcal{E}}[w]$ and can be **twisted** with any vector bundle $\mathcal{V}$ over Σ equipped with an invariant connection D

Twisted pre-D operator $\underline{\overline{\mathfrak{D}}}_A f := \overline{\mathfrak{D}}^{\mathcal{T}}_A f$
Twisted double-D operator $\underline{\overline{\mathbb{D}}}_{AP} f := \overline{\mathbb{D}}^{\mathcal{T}}_{AP} f$
Twisted Thomas-D operator $\underline{\overline{\mathrm{D}}}_A f := \overline{\mathrm{D}}^{\mathcal{T}}_A f$
Explicitly:

$$\underline{\overline{\mathrm{D}}}_A f \overset{g}{=} (m + 2w - 3)\left(\bar{Y}_A w f + \bar{Z}_A{}^a \underline{\nabla}_a f\right) - \bar{X}_A(\mathbf{g}^{ab}\underline{\nabla}_a \underline{\nabla}_b f + \bar{\mathrm{J}} w f)$$

Hypersurface conformal Weyl invariants

A *(scalar) hypersurface conformal Weyl invariant* is a linear combination of (complete) tractor contractions of the expressions of the following form:

even invariants

- contr $\left(\mathrm{h} \cdot \dots \cdot \mathrm{h} \cdot \mathrm{W}^{(k_1)} \cdot \dots \cdot \mathrm{W}^{(k_r)} \cdot \mathrm{N}^{(l_1)} \cdot \dots \cdot \mathrm{N}^{(l_s)}\right)$

odd invariants

- contr $\left(\eta \cdot \mathrm{h} \cdot \dots \cdot \mathrm{h} \cdot \mathrm{W}^{(k_1)} \cdot \dots \cdot \mathrm{W}^{(k_r)} \cdot \mathrm{N}^{(l_1)} \cdot \dots \cdot \mathrm{N}^{(l_s)}\right)$

where

$$\mathrm{W}^{(k)}_{ABCD,P_1\dots P_k} := \mathrm{D}_{P_1} \dots \mathrm{D}_{P_k} \mathrm{W}_{ABCD}$$

and

$$\begin{cases} \mathrm{N}^{(0)}_{A} := \mathrm{N}_A \\ \mathrm{N}^{(l)}_{A,Q_1\dots Q_l} := \underline{\overline{\mathrm{D}}}_{Q_1} \dots \underline{\overline{\mathrm{D}}}_{Q_l} \mathrm{N}_A \text{ for } l \geq 1 \end{cases}$$

Hypersurface conformal quasi-Weyl invariants

Construction:

Step 1 (for $n > 3$)

a) Form $\mathbb{D}_{PP'} \dots \mathbb{D}_{QQ'} \mathrm{C}_{A'ABCDD'}$ or $\overline{\mathbb{D}}_{PP'} \dots \overline{\mathbb{D}}_{QQ'} \mathbb{N}_{AA'}$, b) contract some tractor indices and c) take its symmetric trace-free part

Here $\mathbb{N}_{AA'} := 2\mathrm{X}_{[A'}\mathrm{Z}_{A]}{}^{a}\mathrm{N}_{a}$

There is a modification of this step for $n = 3$

Step 2

Manipulate to $X_{(I} \dots X_J Q_{K\dots)\circ}$

Step 3

"Eliminate" some indices, e.g. $\overline{\mathrm{D}}^{K} Q_{K\dots}$

Definition

Linear combinations of tractor contractions so obtained tractors (or densities) are called *h. c. quasi-Weyl invariants*

The Willmore invariant revisited

Explicitly, using $n = \dim \Sigma$,

$$\overline{\mathrm{D}}_A \mathrm{N}_B = (n-2)\mathrm{Z}_A{}^a \mathrm{Z}_B{}^b \mathring{\mathbf{L}}_{ab} - 2\tfrac{n-2}{n-1}\mathrm{Z}_{(A}{}^a \mathrm{X}_{B)} \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + \mathrm{X}_A \mathrm{X}_B \mathscr{W} + \mathrm{X}_A \mathrm{N}_B |\mathring{\mathbf{L}}|^2$$

where

$$\mathscr{W} = \tfrac{1}{n-1}\overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + \mathrm{P}^{ab}\mathring{\mathbf{L}}_{ab} + \mathbf{H}\,|\mathring{\mathbf{L}}|^2$$

In $n = 2$ it turns out that $\mathrm{P}_{(ab)\circ} = \mathrm{Ric}_{(ab)\circ}$, and we recover the Willmore invariant!

Notice that $\mathrm{X}_A |\mathring{\mathbf{L}}|^2 = \mathrm{N}^B \overline{\mathrm{D}}_A \mathrm{N}_B$, so that $|\mathring{\mathbf{L}}|^2$ appears to be the projecting part of a hypersurface Weyl invariant in any dimension $n \geq 2$.

Conjectures and questions

Conjecture

Almost all hypersurface conformal invariants can be obtained using the quasi-Weyl construction, described above.

Question

Can we find higher dimensional analogues of the Willmore invariant using these methods?
(We have some initial progress in this direction)

Question

Is there any analogue of the Deser-Schwimmer conjecture (Alexakis's theorem) for the global hypersurface conformal invariants?

References

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