A bijection in cohomology of filiform Lie algebras over $\mathbb{Z}_2$

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   $[e_i, e_j] = (i - j)e_{i+j}, i, j \geq 1.$

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	$\mathfrak{m}_0$	$\mathfrak{m}_0(n)$	$\mathfrak{m}_2$	$\mathfrak{m}_2(n)$
$\mathbb{R}$	Done	Done	Done	Open
$\mathbb{Z}_2$	Done	Open	Done	Open

The bijection

Definition

Let n be an integer. We define $\mathfrak{m}_0(n)$: $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$.
$\mathfrak{m}_2(n)$: $[e_1, e_i] = e_{i+1}$, for $2 \leq i \leq n-1$ and $[e_2, e_j] = e_{j+2}$ for $3 \leq j \leq n-2$.

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D_1: $D_1(e^1) = D_1(e^2) = 0, \;\; D_1(e^i) = e^{i-1}, i \geq 3$.
D_2: $D_2(e^i) = 0, i \leq 4, \;\; D_2(e^i) = e^{i-2}, i \geq 5$.
We extend these maps by $D_i(\xi \wedge \zeta) = D_i(\xi) \wedge \zeta + \xi \wedge D_i(\zeta)$ for $i \in \{1, 2\}$.

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- $d_1 = e^1 \wedge D_1$.

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Remarks

- $d_1 = e^1 \wedge D_1$.
- $d_2 = e^1 \wedge D_1 + e^2 \wedge D_2$.
- D_1^2 is a derivation.

Lemma 1

Let $\omega \in \Lambda^k(e^1, \ldots, e^n)$*, then* $e^2 \wedge D_2(\omega) = e^2 \wedge D_1^2(\omega)$.

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Proof

Observe that

$$D_2(e^k) = \begin{cases} D_1^2(e^k) + e^2 & \text{if } k = 4, \\ D_1^2(e^k) & \text{otherwise.} \end{cases}$$

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Write $\omega = e^4 \wedge \omega' + \omega''$ where ω' and ω'' do not contain e^4. Calculate:

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Write $\omega = e^4 \wedge \omega' + \omega''$ where ω' and ω'' do not contain e^4. Calculate:

$$e^2 \wedge D_2(\omega) = e^2 \wedge (e^4 \wedge D_2(\omega') + D_2(\omega''))$$
$$e^2 \wedge D_1^2(\omega) = e^2 \wedge (e^2 \wedge \omega' + e^4 \wedge D_1^2(\omega') + D_1^2(\omega'')) = e^2 \wedge (e^4 \wedge D_1^2(\omega') + D_1^2(\omega'')).$$

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Let $\omega \in \Lambda^k(e^1, \ldots, e^n)$*, then* $e^2 \wedge D_2(\omega) = e^2 \wedge D_1^2(\omega)$*.*

Lemma 2

Let $\omega = e^1 \wedge x + e^2 \wedge y + z$ *with* $x \in \Lambda^{k-1}(e^2, \ldots, e^n)$*,* $y \in \Lambda^{k-1}(e^3, \ldots, e^n)$ *and* $z \in \Lambda^k(e^3, \ldots, e^n)$ *such that* $d_1(\omega) = 0$*. Then* $e^2 \wedge D_2(z) = 0$*.*

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Proof

As ω is a d_1-cocycle, we have that $0 = d_1(\omega) = e^1 \wedge D_1(e^2 \wedge y + z)$. Furthermore, y, z contain no e^1 terms thus we have that $D_1(e^2 \wedge y + z) = 0$ and also

$$0 = e^2 \wedge D_1^2(e^2 \wedge y + z) = e^2 \wedge (e^2 \wedge D_1^2(y) + D_1^2(z)) = e^2 \wedge D_1^2(z).$$

Using the previous lemma, we get the claim.

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Let $\omega = e^1 \wedge x + e^2 \wedge y + z$ *with* $x \in \Lambda^{k-1}(e^2, \ldots, e^n)$, $y \in \Lambda^{k-1}(e^3, \ldots, e^n)$ *and* $z \in \Lambda^k(e^3, \ldots, e^n)$ *such that* $d_1(\omega) = 0$. *Then* $e^2 \wedge D_2(z) = 0$.

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$\dim(Z_k(\mathfrak{m}_2(n))) = \dim(Z_k(\mathfrak{m}_0(n)))$ *for* $2 \leq k \leq n-1$.

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If ω is a cocycle in $\mathfrak{m}_2$ we have that
$0 = d_2(\omega) = e^1 \wedge e^2 \wedge D_2(x) + e^1 \wedge e^2 \wedge D_1(y) + e^1 \wedge D_1(z) + e^2 \wedge D_2(z)$.

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For the inverse assume that ω is a cocycle in $\mathfrak{m}_0$ and use Lemma 2.

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Let $\omega \in \Lambda^k(e^1, \dots, e^n)$*, then* $e^2 \wedge D_2(\omega) = e^2 \wedge D_1^2(\omega)$*.*

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If ω is a cocycle in $\mathfrak{m}_2$ we have that

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$d_1(f(\omega)) = e^1 \wedge e^2 \wedge (D_1(y) + D_1^2(x)) + e^1 \wedge D_1(z)$.
For the inverse assume that ω is a cocycle in $\mathfrak{m}_0$ and use Lemma 2.
Last step, see that f is an involution: $f(f(\omega)) = \omega$.

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Theorem (Cairns-Nikolayevsky-Tsartsaflis, 2014)

For $n \in \mathbb{N}$*, the Lie algebras* $\mathfrak{m}_0(n)$ *and* $\mathfrak{m}_2(n)$ *have the same Betti numbers.*

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Theorem (Cairns-Nikolayevsky-Tsartsaflis, 2014)

For $n \in \mathbb{N}$, the Lie algebras $\mathfrak{m}_0(n)$ and $\mathfrak{m}_2(n)$ have the same Betti numbers.

Proof

We know all the 1-forms and n-forms.

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For $n \in \mathbb{N}$*, the Lie algebras* $\mathfrak{m}_0(n)$ *and* $\mathfrak{m}_2(n)$ *have the same Betti numbers.*

Proof

We know all the 1-forms and n-forms.
Use the formula $b_k = \dim(Z_k(\mathfrak{g})) + \dim(Z_{k-1}(\mathfrak{g})) - \binom{n}{k-1}$.

Thanks

Thank you for your attention!