A Computational Method for Classifying Real Nilpotent Lie Algebras

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AMSI Workshop Differential Geometry, Complex Analysis and Lie Theory, Melbourne, December 5 – 7, 2014

Outline

Motivation

Original (geometric) problem

Classify all real nilpotent Lie algebras in dimension eight which admit soliton inner products whose soliton derivation is simple.

(We say a linear map is simple if its eigenvalues are distinct.)

Motivation

Second (algebraic) problem

Classify all real nilpotent Lie algebras in dimension eight whose "Nikolayevsky derivation" is simple with positive eigenvalues.

Metric Lie algebras

Definition

A metric Lie algebra is a Lie algebra $\mathfrak{g}$ endowed with an inner product Q.

There is a one-to-one correspondence between simply connected Lie groups endowed with left-invariant metrics and metric Lie algebras.

$$(\mathfrak{g}, Q) \qquad\qquad (G, g)$$
$$\mathfrak{g} = \mathrm{Lie}(G) \qquad\qquad T_e G = \mathfrak{g}$$
$$Q = g|_{T_e G} \qquad\qquad g(h) = (l_h)_* Q$$

Geometry of metric Lie algebras

Definition

For a metric Lie algebra $(\mathfrak{g}, Q)$, let the Ricci endomorphism $\mathrm{Ric} : \mathfrak{g} \to \mathfrak{g}$ be the Ricci endomorphism for (G, g), restricted to $T_e G \cong \mathfrak{g}$.

Definition

An inner product on a metric Lie algebra is Einstein if $\mathrm{Ric} = \beta \, \mathrm{Id}$.

Theorem (Jensen, 1969)

Nonabelian nilpotent Lie algebras do not admit Einstein inner products.

Algebraic solitons

Definition

A metric Lie algebra $(\mathfrak{g}, Q)$ is an soliton if the Ricci endomorphism $\mathrm{Ric} : \mathfrak{g} \to \mathfrak{g}$ differs from a derivation by a scalar multiple of the identity: there exists $\beta \in \mathbb{R}$ so that $\mathrm{Ric} - \beta Id = \hat{D}$ is a derivation.

Definition

When $\mathfrak{n}$ is nilpotent, we call an algebraic soliton inner product a nilsoliton inner product. We call the special derivation $\hat{D}$ the soliton derivation.

Interpretation

$(\mathfrak{g}, Q)$ is Einstein at the second level of Lie algebra cohomology.

The three-dimensional Heisenberg algebra

Example

Let $\mathfrak{h}_3$ be the three-dimensional Heisenberg algebra spanned by x, y and z, with $[x, y] = z$. Endow $\mathfrak{h}_3$ with an inner product Q that makes $\mathcal{B} = \{x, y, z\}$ orthonormal. Then

$$[\mathrm{Ric}]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and}$$

$$[\mathrm{Ric}]_{\mathcal{B}} - \left(-\frac{3}{2}\right) Id = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

represents a derivation $\hat{D}$. Hence Q is a nilsoliton inner product with nilsoliton derivation $\hat{D}$.

Philosophy

- Einstein metrics are considered "preferred" metrics on manifolds (canonical, maximally symmetric, extremal for geometric functionals, useful for calculations,...)
- Soliton metrics are generalizations of Einstein metrics:
  - Einstein metrics are soliton
  - soliton metrics are preferred metrics in the absence of Einstein metrics
- Soliton inner products are preferred inner products on Lie algebras (canonical, maximally symmetric, extremal for algebraic functionals?, useful for calculations,...)

Dimension 7 and less

Theorem (Lauret, 2003; Will, 2006)

All nilpotent Lie algebras of dimension six and less admit soliton inner products.

Theorem (Fernández Culma, 2014)

Complete analysis of existence/nonexistence of soliton inner products in dimension 7

In dimension seven:

- Not all $\mathbb{N}$-graded nilpotent Lie algebras soliton inner products
- There are continuous families of nilpotent Lie algebras that admit nilsoliton inner products and there are continuous families of nilpotent Lie algebras that do not admit nilsoliton inner products.

General theory of soliton metrics on nilmanifolds

- Heber, 1998: Einstein inner products on solvable Lie algebras
- Lauret, 2001: soliton inner products on nilpotent Lie algebras
- Geometric invariant theory: Heber, Lauret, Eberlein, Nikolayevsky, Jablonski, ...
- Special classes: Lauret-Will, Nikolayevsky, Eberlein, Arroyo, Oscari, ...
- Simple criterion for existence of soliton inner product: P., 2010; Nikolayevsky, 2011

Real nonabelian nilpotent Lie algebras in dimension 6 and less

Classification in low dimensions

- Finitely many nonabelian nilpotent Lie algebras in each dimension:
  - $\mathfrak{h}_3$ in dimension three
  - $\mathfrak{h}_3 \oplus \mathbb{R}$ and $\mathfrak{l}_4$ in dimension four
  - 8 in dimension five
  - 33 in dimension six
- All of these admit soliton inner products (Lauret, 2003; Will, 2006)

Umlauf, Dixmier, Morozov, Shedler, Vergne, Skjelbred and Sund, Beck, Kolman, Nielsen, De Graaf.

Nonabelian real nilpotent Lie algebras in dimension 7

Theorem (Gong, 1998)

In dimension 7, the moduli space of real nilpotent Lie algebras consists of 140 isolated nilpotent Lie algebras and 9 curves of nonisomorphic Lie algebras.

Safiullina, Romdhani, Seeley, Ancochea and Goze, Carles...

Caution

In all dimensions and over any fields, classification problems like this are highly prone to error (of both omission and repetition).

The algebraic problem: Classification in dimensions 8 and higher

Moduli spaces are huge

In dimensions 8 and higher, the moduli space of nilpotent Lie algebras is very large.

- There are huge continuous families of nonisomorphic Lie algebras.
- There are always families of characteristically nilpotent Lie algebras ([Hakimdjanov, 1991]).
- No general classification exists, although there are classifications of subclasses over various fields.

The Nikolayevsky derivation

Originally defined in context of Einstein solvable metric Lie algebras; but actually a purely algebraic object.

Definition

Let $\mathfrak{g}$ be a nilpotent Lie algebra. The Nikolayevsky derivation is the derivation so that

$$\operatorname{trace} D^N \circ F = \operatorname{trace} F$$

for all derivations F of $\mathfrak{g}$.

Example

Let $\mathfrak{h}_3$ be the three-dimensional Heisenberg Lie algebra. Then, relative to the basis $\{x, y, z\}$,

$$D^N = \tfrac{2}{3}\operatorname{diag}(1, 1, 2) = \tfrac{2}{3}\hat{D}.$$

More about the Nikoalyevsky derivation

Theorem (Nikoalyevsky, 2011)

The Nikolayevsky derivation has the following properties:

1. *D^N is unique up to automorphism.*
2. *If $\mathfrak{n}$ admits a soliton inner product, then D^N is a positive scalar multiple of $\hat{D}$.*
3. *Eigenvalues are rational.*

It may be that $D^N \equiv 0$. Often D^N has repeated eigenvalues; e.g. if $\mathfrak{n}$ is two-step. However, if $\mathfrak{n}$ is higher step, D^N is often simple– always if $\mathfrak{n}$ is filiform (?).

The Nikoalyevsky derivation

The Nikolayevsky derivation is a simple and useful algebraic invariant.

Example

Label in Seeley, 1993	Eigenvalues of D^N (rescaled and ordered)
$(2,4,7)_A$	$(1,4,4,5,5,6,6)$
$(2,4,7)_B$	$(6,11,15,17,21,27,28)$
$(2,4,7)_C$	$(11,20,29,31,40,42,51)$
$(2,4,7)_D$	$(7,10,12,17,19,24,29)$
$(2,4,7)_E$	$(3,5,5,8,8,11,13)$
$(2,4,7)_F$	$(5,5,6,11,11,16,16)$
$(2,4,7)_G$	$(20,21,22,42,43,62,64)$
$(2,4,7)_H$	$(1,1,1,2,2,3,3)$
$(2,4,7)_I$	$(7,10,11,17,21,24,28)$
$(2,4,7)_J$	$(7,7,10,14,17,21,24)$
$(2,4,7)_K$	$(5,6,7,11,12,17,18)$
$(2,4,7)_L$	$(5,10,11,15,16,20,21)$
$(2,4,7)_M$	$(11,22,30,33,41,52,55)$
$(2,4,7)_N$	$(15,19,23,34,38,42,53)$
$(2,4,7)_O$	$(2,3,4,5,6,7,8)$
$(2,4,7)_P$	$(1,1,1,2,2,2,3)$
$(2,4,7)_Q$	$(3,5,6,8,9,11,14)$
$(2,4,7)_R$	$(1,2,2,3,3,4,5)$

Table : Indecomposable nilpotent Lie algebras of type $(2,4,7)$ and the eigenvalue types of D^N.

The answer to the geometric problem

Theorem (Kadioglu-P. 2013; P., in preparation)

Complete classification of nilpotent Lie algebras in dimensions 7 and 8 for which the Nikolayevsky derivation is simple.

- *Dimension 7:* $4 + 11$ *nonsoliton;* $29 + 23$ *soliton (no continuous families)*
- *Dimension 8:* $50 + 57$ *nonsoliton;* $109 + \approx 141$ *soliton (continuous families)*

- List is complete but list still needs to be sorted into connected pieces
- Does not rely on previous classification of nilpotent Lie algebras
- Uses the algebraic theorem; both use MATLAB
- There are new phenomena in dimension 8
- First half were tested for stability by Jablonski, Petersen, Williams

Index sets; stratification of $\bigwedge^2(\mathbb{R}^*) \otimes \mathbb{R}$

Let

$$\Upsilon_n = \{(i, j, k) : 1 \leq i < j \leq n, 1 \leq k \leq n\}.$$

A Lie bracket $\mu \in \bigwedge^2(\mathbb{R}^*) \otimes \mathbb{R}$ can be written as

$$\mu = \sum_{(i,j,k) \in \Upsilon_n} \mu_{ij}^k (e_i^* \wedge e_j^*) \otimes e_k.$$

Definition

For $\Lambda \subseteq \Upsilon_n$, let

$$\mathcal{S}_\Lambda = \{\mu : \mu_{ij}^k \neq 0 \text{ for } i < j \Leftrightarrow (i, j, k) \in \Lambda\}.$$

Then $\bigwedge^2(\mathbb{R}^*) \otimes \mathbb{R} = \bigcup_{\Lambda \subseteq \Upsilon_n} \mathcal{S}_\Lambda$.

Canonical framing

Suppose that $\mathfrak{n}$ is a nilpotent Lie algebra such that D^N is simple with positive eigenvalues. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be the eigenvalues, and let $e_1, e_2, \ldots, e_n$ be the associated eigenvectors. Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ be the associated eigenspaces. Because D^N is a derivation, $[\mathfrak{n}_{\lambda_i}, \mathfrak{n}_{\lambda_j}] \subseteq \mathfrak{n}_{\lambda_i + \lambda_j}$. Then the nonzero structure constants for $\mathfrak{n}$ all lie in

$$\Theta_n = \{(i, j, k) \, : \, 1 \leq i < j < k \leq n\}.$$

Therefore, $\mathfrak{n}$ is in $\mathcal{S}_\Lambda$, where $\Lambda \subseteq \Theta_n$.

Dimension four

$$\Theta_4 = \{(1,2,3),(1,2,4),(1,3,4),(2,3,4)\}$$

Example

Let $\Lambda = \{(1,2,3),(1,3,4)\}$. Then $\mathcal{S}_\Lambda$ is the set of all algebras of form

$$[e_1, e_2] = \mu_{12}^3 e_3, [e_1, e_3] = \mu_{13}^4 e_4, \text{with } \mu_{12}^3, \mu_{13}^4 \neq 0.$$

Example

Let $\Lambda = \{(1,2,3),(1,2,4)\}$. If $\{e_i\}$ is a canonical framing for simple D^N with positive eigenvalues, then

$$\lambda_1 + \lambda_2 = \lambda_3 \quad \text{and} \quad \lambda_1 + \lambda_2 = \lambda_4 \Rightarrow \lambda_3 = \lambda_4.$$

If $\{e_i\}$ is a canonical framing, D^N is not simple.

Y and U

For an index set Λ, define matrices Y and U so that rows of Y are $e_i + e_j - e_k$, $(i, j, k) \in \Lambda$, and $U = YY^T$. Let the matrix $\hat{Y}$ be the same but over $\mathbb{Z}_2$.

Example

Let $\Lambda = \{(1,2,3),(1,3,4)\}$. The matrices Y, $\hat{Y}$ and U are

$$Y = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}, \hat{Y} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Theorem (Payne, 2010)

$A = 2U - 4[1]$ is a generalized Cartan matrix. If of finite or affine type, all Lie algebras in $\mathcal{S}_\Lambda$ admit soliton inner products.

Inadmissible U

Example

Let $\Lambda = \{(1,2,3),(1,2,4)\}$.
The matrices Y, $\hat{Y}$ and U are

$$Y = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}, \hat{Y} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Pruning criterion

If D^N is simple, U can not have any entries of 2.

Computing D^N

Fix Λ.

- All Lie algebras in $\mathcal{S}_\Lambda$ have the same Nikolayevsky derivation.
- Either all Lie algebras in $\mathcal{S}_\Lambda$ admit a soliton inner product or none admit a soliton inner product.

Lemma (Payne, 2009)

Fix a nonempty index set $\Lambda \subseteq [n]^3$ containing no elements of the form (i,j,j) or (i,j,i). For all choices of $\mathbf{b}$ in the solution space to $U\mathbf{v} = [1]_{m\times 1}$, the vector $\mathbf{v}_{D^N}$ defined by $\mathbf{v}_{D^N} = -\mathbf{b}^T Y + [1]_{1\times n}$ is independent of $\mathbf{b}$. Let D^N be the endomorphism of $\mathbb{R}^n$ given with respect to the basis $\mathcal{B}$ by the diagonal matrix $[D^N]_{\mathcal{B}} = \mathrm{diag}(\mathbf{v}_{D^N})$. Then D^N is the Nikolayevsky derivation for all Lie algebras in the family $\mathcal{F}_\Lambda$.

Theorem (Payne, 2010; Nikolayevsky, 2011)

A Lie algebra $\mathfrak{n}$ with simple D^N admits a soliton inner product if and only if $U\mathbf{v} = [1]$ has a solution $\mathbf{v}$ with all entries positive.

MATLAB algorithm (simplified)

Hypotheses: D^N is simple with positive eigenvalues, $\{e_i\}$ is the canonical framing.

1. Enumerate all possible index sets $\Lambda \subseteq [n]^3$ such that elements (i, j, k) of Λ satisfy $i < j < k$.
2. For each such Λ,
   a. Compute Y and U.
   b. Prune: Check for combinatorial properties of Λ that imply that no elements of $\mathcal{S}_\Lambda$ satisfy the Jacobi Identity or that D^N is not simple. If those conditions hold, move to the next Λ in the list. Otherwise, proceed to c.
   c. Use Y to find the Nikolayevsky derivation D^N associated with Λ.
   d. If the eigenvalues of D^N are not distinct, positive and in ascending order, stop and go to the next Λ in the list.
   e. Store Λ.

Lemma

Let $\mathfrak{n}$ be a nilpotent Lie algebra. If $\{e_i\}$ is a canonical framing, and D^N is simple with positive eigenvalues, then the set

$$\Lambda = \{(i,j,k) \in \Theta_n \,:\, \mu_{ij}^k \neq 0\}$$

is an isomorphism invariant of $\mathfrak{n}$.

Therefore, if $\mathfrak{n}$ satisfies our hypotheses, there will only be one Λ in our list so that $\mathfrak{n}$ is in $\mathcal{S}_\Lambda$.

Question

For fixed Λ, how can we parametrize isomorphism classes of Lie algebras in $\mathcal{S}_\Lambda$?

Isomorphism classes

- Since D^N is simple, isomorphisms between elements of $\mathcal{S}_\Lambda$ rescale basis vectors in the canonical framing. Therefore, with respect to the canonical framing, isomorphism classes in $\bigwedge^2(\mathbb{R}^*) \otimes \mathbb{R}$ are orbits of the diagonal subgroup of $GL_n(\mathbb{R})$, where

$$(g\mu)(\cdot, \cdot) = g\mu(g^{-1}\cdot, g^{-1}\cdot)$$

  for $g \in GL_n(\mathbb{R})$ and $\mu \in \bigwedge^2(\mathbb{R}^*) \otimes \mathbb{R}$.

- We will encode the action of the diagonal subgroup of $GL_n(\mathbb{R})$ on $\bigwedge^2(\mathbb{R}^*) \otimes \mathbb{R}$ (with respect to the canonical framing) as the product of actions $\rho_Y \times \rho_{\hat{Y}}$, using the isomorphism $\mathbb{R} \times \mathbb{Z}_2 \cong \mathbb{R}$ with $(\lambda, \pm 1) \sim \pm e^\lambda$.

Definition of actions

Definition

Define the action $\rho_Y : \mathbb{R}^n \times \mathbb{R}^{|\Lambda|} \to \mathbb{R}^{|\Lambda|}$ by

$$\mathbf{d} \star \mathbf{z} \mapsto \mathbf{z} + Y\mathbf{d}$$

and define the action $\rho_{\hat{Y}} : \mathbb{Z}_2^n \times \mathbb{Z}_2^{|\Lambda|} \to \mathbb{Z}_2^{|\Lambda|}$ by

$$\hat{\mathbf{d}} \star \hat{\mathbf{z}} \mapsto \hat{\mathbf{z}} + \hat{Y}\hat{\mathbf{d}}.$$

Note: Orbits are translates of column spaces of Y and $\hat{Y}$.

Isomorphism classes are translates of column spaces

Theorem (P., 2012)

Let $\Lambda \subseteq \Theta_n$. Let $\mathfrak{n}_\mu, \mathfrak{n}_\nu$ be Lie algebras in $\mathcal{S}_\Lambda$ which are isomorphic by ϕ of form $\phi(e_i) = c_i e_i$. Then $\mathfrak{n}_\mu$ and $\mathfrak{n}_\nu$ are isomorphic if and only if

- $[\ln|\mu_{ij}^k|]_{(i,j,k)\in\Lambda}$ *and* $[\ln|\nu_{ij}^k|]_{(i,j,k)\in\Lambda}$ *are in the same orbit of the* ρ_Y *action, and*
- $[\operatorname{sgn}|\mu_{ij}^k|]_{(i,j,k)\in\Lambda}$ *and* $[\operatorname{sgn}|\nu_{ij}^k|]_{(i,j,k)\in\Lambda}$ *are in the same orbit of the* $\rho_{\hat{Y}}$ *action.*

Examples

Easy example

For $\Lambda = \{(1,2,3),(1,3,4)\}$, $\text{rank}_{\mathbb{R}}\, Y = 2$ and $\text{rank}_{\mathbb{Z}_2}\, \hat{Y} = 2$. Hence orbits are $\mathbb{R}^2$ and $\mathbb{Z}_2^2$. A transversal for for ρ_Y is $\{(0,0)\} \subseteq \mathbb{R}^2$ and a transversal for for $\rho_{\hat{Y}}$ is $\{(0,0)\} \subseteq \mathbb{Z}_2^2$, giving

$$(\mu_{12}^3, \mu_{13}^4) = (+e^0, +e^0) = (1,1).$$

Therefore all Lie algebras in $\mathcal{S}_\Lambda$ are isomorphic to the one with

$$[x_1, x_2] = x_3 \qquad [x_1, x_3] = x_4.$$

Examples

Easiest hard example

For $\Lambda = \{(1,2,4),(1,3,5),(1,5,6),(2,4,6),(2,5,7),(3,4,7)\}$, $\mathrm{rank}_{\mathbb{R}}\, Y = 5$ and $\mathrm{rank}_{\mathbb{Z}_2}\, \hat{Y} = 5$. Hence ρ_Y orbits $\mathbf{a} + \mathrm{col}(Y) \subseteq \mathbb{R}^6$ and $\rho_{\hat{Y}}$ orbits $\hat{\mathbf{a}} + \mathrm{col}(\hat{Y}) \subseteq \mathbb{Z}_2^6$ are codimension one. Transversals: Use log of $([1] + \ker Y^T) \cap (\mathbb{R}^+)^6$ for ρ_Y and $\{(0,0,0,0,0,0),(0,0,0,0,0,1)\}$ for $\rho_{\hat{Y}}$ to get

$$(\mu_{12}^4, \mu_{13}^5, \mu_{15}^6, \mu_{24}^6, \mu_{25}^7, \mu_{34}^7) = (1+t, 1-t, 1, 1, 1-t, \pm(1+t))$$

where $-1 < t < 1$. Still need to solve the Jacobi Identity on this subset of $\mathcal{S}_\Lambda$.

Idea of proof

Suppose we apply an isomorphism of form $\phi(e_i) = e^{c_i} e_i$. How do $[\ln |\mu_{ij}^k|]_{(i,j,k)\in\Lambda}$ and $[\mathrm{sgn}\, |\mu_{ij}^k|]_{(i,j,k)\in\Lambda}$ change?
If $[e_i, e_j] = \sum \mu_{ij}^k e_k$, then

$$[c_i e_i, c_j e_j] = \sum \left(\frac{c_i c_j}{c_k} \mu_{ij}^k \right) c_k e_k.$$

If $c_i = \mathrm{sgn}(c_i) e^{\ln |c_i|}, c_j = \mathrm{sgn}(c_j) e^{\ln |c_j|}, c_k = \mathrm{sgn}(c_k) e^{\ln |c_k|}$,

$$\ln |\nu_{ij}^k| = \ln |c_i| + \ln |c_j| - \ln |c_k| + \ln |\mu_{ij}^k|$$

$$\ln |\nu_{ij}^k| = \ln |\mu_{ij}^k| + (e_i + e_j - e_k)(\ln |c_1|, \ldots, \ln |c_n|)^T$$

and (additively)

$$\mathrm{sgn}(\nu_{ij}^k) = \mathrm{sgn}(c_i) + \mathrm{sgn}(c_j) - \mathrm{sgn}(c_k) + \mathrm{sgn}(\mu_{ij}^k).$$

Parametrizing continuous families

Goal

Find a compact, semi-algebraic set $S_\Lambda \subseteq \mathcal{S}_\Lambda$ so that each isomorphism class is represented exactly once in S_Λ.

1. First find a transversal for the action of Y on $\mathcal{S}_\Lambda$
2. Then solve the Jacobi Identity

Finding the transversal

- Find a set to use as transversal. There are different choices. It's harder to get a bounded transversal.
- Show that it is a transversal by showing that projection along ρ_Y orbits to ker Y^T is a diffeomorphism. To do this, use Hadamard's Global Inverse Function Theorem. This requires computing the Jacobian J.
- To get a diffeomorphism, a criterion for the nonsingularly of J can be found that depends only on the combinatorics of Λ.

Jacobi Identity

Definition

A pair of triples in Λ *overlaps* if the inner product of the corresponding vectors is -1. A pair of overlapping triples can be assigned a multiplicative sign. A pair of overlapping triples determines a unique *quadruple* (i, j, k, m).

The quadruple (i, j, k, m) corresponds to the equation

$$\langle [[e_i, e_j], e_k] + [[e_k, e_i], e_j] + [[e_j, e_k], e_i], e_m \rangle = 0.$$

and each pair of triples with that quadruple gives a nonzero term. For example, the first term comes from (i, j, l) and (l, k, m).

Jacobi Identity

Theorem (P., 2014)

Let $\Lambda \subseteq \Theta_n$ be an index set and let

$$\mathbf{t}_1 = (i_1, j_1, k_1),\ \mathbf{t}_2 = (i_2, j_2, k_2), \ldots,\ \mathbf{t}_m = (i_m, j_m, k_m)$$

be an enumeration of the triples in Λ. Let Q be the set of quadruples for Λ. The Jacobi Identity for elements of $\mathcal{S}_\Lambda(K)$ is equivalent to the system of equations

$$\sum_{q(\mathbf{t}_p, \mathbf{t}_r) = (i,j,k,m)} \operatorname{sgn}(\mathbf{t}_p, \mathbf{t}_r) \alpha_{i_p j_p}^{k_p} \alpha_{i_r j_r}^{k_r} = 0, \quad (i,j,k,m) \in Q. \qquad (1)$$

Pruning criterion

Each quadruple for Λ must have multiplicity at least two.

Solving the Jacobi Identity

Good basis

Before attempting to solve the Jacobi Identity, find a good basis for parametrizing the transversal.

There are few cases to consider!

It turns out that the Jacobi Identity for $\mathcal{S}_{\Lambda_1}$ and the Jacobi Identity for $\mathcal{S}_{\Lambda_2}$ are *equivalent* if the combinatorics of quadruples in Λ_1 and Λ_2 are the same. Virtually all examples fall into just four families.

Example

For $\Lambda = \{(1,2,4),(1,3,5),(1,5,6),(2,4,6),(2,5,7),(3,4,7)\}$, there are only two pairs of overlapping triples and one quadruple:

$$q((1,2,4),(3,4,7)) = q((1,3,5),(2,5,7)) = (1,2,3,7).$$

The Jacobi Identity is $\mu_{12}^4\mu_{34}^7 - \mu_{13}^5\mu_{25}^7 = 0$.
We already found the transversal
$(\mu_{12}^4, \mu_{13}^5, \mu_{15}^6, \mu_{24}^6, \mu_{25}^7, \mu_{34}^7) = (1+t, 1-t, 1, 1, 1-t, \pm(1+t))$
Solving the Jacobi Identity, we get

$$(1+t)(1+t) - (1-t)(1-t) = 0 \Rightarrow t = 0$$

$$-(1+t)(1+t) - (1-t)(1-t) = 0 \Rightarrow \text{no solutions}$$

Thus the Lie algebras in $\mathcal{S}_\Lambda$, up to isomorphism, are

$$(\mu_{12}^3, \mu_{13}^5, \mu_{15}^6, \mu_{24}^6, \mu_{25}^7, \mu_{34}^7) = (1,1,1,1,1,1).$$

Note

- In the previous example, the transversal $(\mu_{12}^4, \mu_{13}^5, \mu_{15}^6, \mu_{24}^6, \mu_{25}^7, \mu_{34}^7) = (1+t, 1-t, 1, 1, 1-t, \pm(1+t))$ came from the vector $e_1 + e_6 - e_2 - e_4$ in $\ker(Y^T)$. The same quadruple $(1, 6; 2, 4)$ encoded the Jacobi Identity.
- All other examples (about 60 of them) of Λ having one quad of multiplicity two yield exactly the same computation.

The quadruples for Λ define vectors $e_1 + e_6 - e_2 - e_4$ in $\ker Y^T$. Almost always they span $\ker(Y^T)$.

The key to solving the Jacobi Identity for hundreds of classes $\mathcal{S}_\Lambda$

The computation of the Jacobi Identity falls into less than 10 types, combinatorially. Almost all fall into just five types.

Theorem (P., in preparation)

Let $\Lambda \subseteq \Theta_n$ be an index set that is null space spanning. Then

1. *If Λ has exactly one quadruple of multiplicity two, then $\widetilde{\mathcal{L}}_\Lambda(\mathbb{R})$ is finite.*
2. *If Λ has exactly two quadruples of multiplicity two, then $\widetilde{\mathcal{L}}_\Lambda(\mathbb{R})$ is finite.*
3. *If Λ has just one quadruple of multiplicity three, then $\widetilde{\mathcal{L}}_\Lambda(\mathbb{R})$ is one-dimensional.*
4. *If Λ has one quadruple of multiplicity three, and one two quadruple of multiplicity two, and the quadruples have exactly one common triple, then $\widetilde{\mathcal{L}}_\Lambda(\mathbb{R})$ is one-dimensional.*

Virtually all examples are "null-space spanning."

Questions

Problem

Classify all eight-dimensional nilpotent Lie algebras admitting some derivation with positive eigenvalues. Use weight spaces instead of D^N.

Problem

Find all degenerations of the eight-dimensional nilpotent Lie algebras so that D^N is simple with positive eigenvalues.

Problem

Classify higher-dimensional nilpotent Lie algebras so that D^N has positive eigenvalues and other restrictions hold (filiform, quasi-filiform, two-step, extra geometric structure, ...)

Questions

Problem

Refine the definition of D^N to get finer invariants when the rank of the Lie algebra is large.

Problem

Is there a preferred inner product when no soliton inner product exists? Say, when D^N is nonsingular?

Problem

How does the combinatorics at play here relate to Lie algebra cohomology?

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