Willmore–like functionals for surfaces in 3–dimensional Thurston geometries

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Outline

1. Willmore functional
2. Bianchi–Cartan–Vranceanu family of Riemmannian manifolds
3. The CMC spheres in these family of manifolds
4. Willmore–like functional for which the CMC spheres are critical points

Willmore functional

Let M be a closed orientable surface and $f\colon M \to N$ be an immersion of M into a 3–dimensional Riemannian manifold N. The Willmore functional has the form:

$$\mathcal{W}(f) = \int_M \left(H^2 + \overline{K}\right) d\mu.$$

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Fact 2.

In the class of immersions of a 2–sphere into $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ the minimum of the functional $\mathcal{W}(f)$ is attained exactly at the round spheres.

The family of Riemannian manifolds $E(k,\tau)$

If $k \geqslant 0$ then $E(k,\tau)$ is $\mathbb{R}^3$ with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{k}{4}(x^2 + y^2)\right)^2} + \left(dz + \frac{\tau(ydx - xdy)}{1 + \frac{k}{4}(x^2 + y^2)}\right)^2.$$

If $k < 0$ then $E(k,\tau)$ is the product $\mathrm{D}^2(\frac{2}{\sqrt{-k}}) \times \mathbb{R}$ with the same metric, where $\mathrm{D}^2(\frac{2}{\sqrt{-k}}) = \{(x,y) \mid x^2 + y^2 < \frac{4}{-k}\}$.

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The family $E(k,\tau)$ includes four Thurston geometries:

- The product $\mathbb{S}^2 \times \mathbb{R}$: $k > 0, \tau = 0$;
- The product $\mathbb{H}^2 \times \mathbb{R}$: $k < 0, \tau = 0$;
- The Heisenberg group Nil: $k = 0, \tau \neq 0$;
- The Lie group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$: $k < 0, \tau \neq 0$.

The CMC spheres in $E(k, \tau)$

Rotationally invariant CMC spheres

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The projection $E(k,\tau) \to E(k,\tau)/\mathrm{SO}(2)$ is a Riemannian submersion, where the factor–space $E(k,\tau)/\mathrm{SO}(2)$ has the metric:

$$d\widetilde{s}^2 = \frac{1}{(1+\frac{k}{4}u^2)^2}du^2 + \frac{1}{1+\tau^2 u^2}dv^2.$$

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A profile of a rotationally invariant CMC surface satisfies:

$$\begin{cases} \dot{u} = \left(1+\frac{k}{4}u^2\right)\cos\sigma, \\ \dot{v} = \sqrt{1+\tau^2 u^2}\sin\sigma, \\ \dot{\sigma} = 2H - \left(\frac{1}{u} - k\frac{u}{4}\right)\sin\sigma. \end{cases}$$

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The ODE has the following first integral:

$$J = \frac{u}{1 + \frac{k}{4} u^2} \left(\sin \sigma - H u \right).$$

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Claim

- If $k \leqslant 0$, then for any H such that $H^2 > \frac{-k}{4}$ there exists a rotationally invariant CMC sphere of constant mean curvature H in $E(k,\tau)$; moreover, if $H^2 \leqslant \frac{-k}{4}$ then there exists no CMC sphere of constant mean curvature H in $E(k,\tau)$.
- If $k > 0$, then for any $H \neq 0$ there exists a rotationally invariant CMC sphere of constant mean curvature H in $E(k,\tau)$.
- For every rotationally invariant CMC sphere in $E(k,\tau)$ the first integral vanishes: $J = 0$.
- The CMC spheres in $E(k,\tau)$ are unique up to isometries.

Willmore–like functionals

For an immersion $f : M \to E(k,\tau)$ of a closed orientable surface M into $E(k,\tau)$ set:

$$E_{\alpha,\beta}(f) = \int_M \left(H^2 + \alpha\overline{K} + \beta\right) d\mu. \quad (1)$$

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The Euler–Lagrange equation of $E_{\alpha,\beta}(f)$ is:

$$\Delta H + H(2H^2 - 2K + (1-2\alpha)(k-4\tau^2)\nu^2 + \\ + k - 2\beta - \alpha\tau^2) + 2\alpha(k-4\tau^2)\mathrm{div}(\nu T) = 0.$$

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$$\frac{\rho^2 + \tau^2\rho^4}{(1 + \frac{k}{4}\rho^2)^2} d\theta^2 + ds^2.$$

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$$2H^3 + H(2\frac{1 + \frac{k}{4}u^2}{u\sqrt{1+\tau^2u^2}} \frac{d}{ds^2} \frac{u\sqrt{1+\tau^2u^2}}{1+\frac{k}{4}u^2} + \\ + \frac{1}{2}(k - 4\tau^2)\frac{\cos^2\sigma}{1+\tau^2u^2} + \frac{k}{2}) + \\ + \frac{1}{2}(k - 4\tau^2)\frac{1+\frac{k}{4}u^2}{u\sqrt{1+\tau^2u^2}} \frac{d}{ds} \frac{u\cos\sigma\sin\sigma}{(1+\frac{k}{4}u^2)\sqrt{1+\tau^2u^2}} = 0$$

Willmore–like functionals

Theorem

The CMC spheres in $E(k,\tau)$ are critical points of the following Willmore–like functional:

$$E(f) = \mathcal{W}(f) + \int_M \left(-\frac{3}{4}\overline{K} + \frac{k}{4} - \frac{\tau^2}{4}\right) d\mu.$$

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Claim

For the Lie group group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ ($k = -1, \tau = -\frac{1}{2}$) the functional $E(f)$ attains its proper minimum in the class of rotationally invariant spheres exactly at the CMC spheres.

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THANK YOU
FOR
YOUR ATTENTION!